

An Elementary Theory for Beam Waveguides

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An elementary theory for beam waveguides based upon geometrical optics as well as upon the HUYGENS undulatory theory is developed. A homogeneous FREDHOLM integral equation of the first kind is derived. It connects the field distribution as observed on a mathematical surface behind the n -th lens with the given initial distribution. If the spacing between the lenses is chosen such that the beam waveguide operates in a stable condition, then the eigendistributions are periodically reproduced along the beam waveguide. For these eigenmodes a homogeneous FREDHOLM integral equation of the second kind is formulated; the diffraction losses associated with the pq -th eigenmode are determined by the corresponding eigenvalue. As pointed out by HEFFNER, the laser beam may be considered a special superposition of eigenmodes uniformly displaced in the direction of propagation. This superposition yields mode patterns which are periodically reproduced from lens to lens. It is suggested that the laser resonator be matched with its optically analogous beam waveguide for an optimum transmission.

The general theory is applied to a beam waveguide whose lenses are a distance of $d=3f$ apart. It is demonstrated that the theory presented is completely consistent with the classical laws in the zero-wavelength limit.

The recent development of the laser has made an extremely large frequency range available for use in communications. Unfortunately, the utilization of this new frequency spectrum is seriously hampered because the propagation of light through the atmosphere is extremely vulnerable to not only fog, dust, rain or snow, but also to turbulence. To overcome these difficulties one might think of protecting the light path. A long-distance transmission would not, however, be efficient if only an ordinary pipe were used. The beam, even if it had an extremely small divergence, would strike the side walls within a distance of a few hundred meters. EAGLESFIELD¹ has reported the possibility of transmitting light through a pipe of precision bore whose inner surface has a mirror finish. GOUBAU and his co-workers²⁻⁸ have investigated the application of so-called beam waveguides to long-distance transmission of light. The beam waveguide, which has been developed originally for millimeter and submillimeter waves, differs from conventional waveguides in that it consists of a sequence of lenses. The transmission along such a

sequence of lenses is, however, stable only for certain ranges of spacing between the lenses⁹.

In the following paper we shall develop a geometrical theory as well as an undulatory theory for beam waveguides which operate at optical frequencies. As is usual in optics, we apply KIRCHHOFF's approximation to reduce the vector problem to a scalar one by considering only a linearly polarized electromagnetic wave. We elaborate the analogy between a beam waveguide and a resonator system with spherical mirrors. We shall derive a matrix and an integral relation which describe the image properties of a beam waveguide. The matrix equation is based upon geometrical optics and the integral equation upon HUYGENS' undulatory theory. It is demonstrated that this integral equation yields solutions which are consistent with the classical laws and, in particular, that they agree with geometrical optics in the zero-wavelength limit. The analogy as mentioned above suggests a condition for optimum matching of the resonator system to its corresponding beam waveguide.

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¹ C. C. EAGLESFIELD, Proc. Inst. Electr. Engrs. (London) **109 B**, 26 [1962].

² G. GOUBAU and J. R. CHRISTIAN, A New Waveguide for Millimeter Waves, presented at URSI-IRE Fall Meeting, San Diego, Calif., October 1959.

³ G. GOUBAU and F. SCHWERING, On the Guided Propagation of Electromagnetic Wave Beams, URSI-IRE Spring Meeting, Washington, D.C., May 1960.

⁴ G. GOUBAU and F. SCHWERING, Inst. Radio Engrs., Trans. Antennas Propagation **9**, 248 [1961].

⁵ F. SCHWERING, Arch. elektr. Übertragung **15**, 555 [1961].

⁶ J. R. CHRISTIAN and G. GOUBAU, Inst. Radio Engrs., Trans. Antennas Propagation **9**, 256 [1961].

⁷ J. R. CHRISTIAN and G. GOUBAU, Proc. Inst. Radio Engrs. **49**, 1679 [1961].

⁸ J. B. BEYER and E. H. SCHEIBE, Inst. Electr. Electronics Engrs., Trans. Microwave Theory Techn. **11**, 18 [1963].

⁹ J. R. PIERCE, Theory and Design of Electron Beams, D. van Nostrand Company, Inc., New York 1954, Chapter XI, pp. 194—213.



1. Geometrical Theory for a Beam Waveguide

It is assumed that the reader is familiar with the basic laws of geometrical optics^{10, 11}. In this elementary theory only those points and rays will be taken into account which lie in the immediate neighborhood of the axis; terms involving squares and higher powers of off-axis distances, or of the angles which the ray makes with the axis, will be neglected. Figure 1 shows a periodic sequence of thin lenses with equal focal length f a distance d apart. For a beam of light a paraxial path of radius ϱ and slope ϱ' is schematically traced. The radius and slope just to the right of the n -th lens are called ϱ_n and ϱ_n' respectively⁹

$$\begin{pmatrix} \varrho_{n+1} \\ \varrho_{n+1}' \end{pmatrix} = \begin{pmatrix} 1 & d \\ -1/f & (1-d/f) \end{pmatrix} \begin{pmatrix} \varrho_n \\ \varrho_n' \end{pmatrix} = \Gamma \begin{pmatrix} \varrho_n \\ \varrho_n' \end{pmatrix} \quad (1)$$

where Γ represents the matrix. The eigenvalues of Γ are the roots of the secular equation

$$\hat{\gamma}^2 - \hat{\gamma} \text{trace } \Gamma + \det \Gamma = 0 \quad (2a)$$

$$\text{or} \quad \hat{\gamma}^2 - (2 - d/f) \hat{\gamma} + 1 = 0. \quad (2b)$$

The roots are

$$\hat{\gamma}_{\pm} = \frac{1}{2} [\text{trace } \Gamma \pm \sqrt{(\text{trace } \Gamma)^2 - 4}]. \quad (3)$$

If the roots are different, the two eigenvectors are linearly independent and two linearly independent solutions are obtained by identifying the initial values (ϱ_0, ϱ_0') with these eigenvectors

$$\Gamma \begin{pmatrix} \varrho_0^{(\pm)} \\ \varrho_0'^{(\pm)} \end{pmatrix} = \gamma_{\pm} \begin{pmatrix} \varrho_0^{(\pm)} \\ \varrho_0'^{(\pm)} \end{pmatrix}. \quad (4)$$

Considering the beam waveguide as a cascade of lenses we obtain by iteration from (1)

$$\begin{pmatrix} \varrho_n \\ \varrho_n' \end{pmatrix} = \Gamma^n \begin{pmatrix} \varrho_0 \\ \varrho_0' \end{pmatrix} \quad (5)$$

$$\text{and using (4)} \quad \begin{pmatrix} \varrho_n^{(\pm)} \\ \varrho_n'^{(\pm)} \end{pmatrix} = \hat{\gamma}_{\pm}^n \begin{pmatrix} \varrho_0^{(\pm)} \\ \varrho_0'^{(\pm)} \end{pmatrix}. \quad (6)$$

If $|\text{trace } \Gamma| > 2$, i. e., $d > 4f$ or $d/f < 0$ ¹², the roots $\hat{\gamma}_+$ and $\hat{\gamma}_-$ are real and either

$$\lim_{n \rightarrow \infty} |\hat{\gamma}_{\pm}^n| \rightarrow \infty \quad \text{or} \quad \lim_{n \rightarrow -\infty} |\hat{\gamma}_{\pm}^n| \rightarrow \infty.$$

Such solutions of (6) are in conflict with the requirement that the paths must remain finite; they would depart further and further from the axis. Hence acceptable solutions are obtained if and only if

$$|\text{trace } \Gamma| = |2 - d/f| \leq 2 \quad (7)$$

which yields the region of stability or low-loss region in the notation of BOYD and KOGELNIK¹³

$$0 \leq d \leq 4f. \quad (8)$$

For a separation d belonging to this low-loss region we may define a real parameter γ by

$$\hat{\gamma}_+ = e^{i\gamma} \quad \text{and} \quad \hat{\gamma}_- = e^{-i\gamma}. \quad (9)$$

A path oscillates periodically along the beam waveguide with a periodicity

$$N = \frac{2\pi}{\arccos(\frac{1}{2} \text{trace } \Gamma)} = \frac{2\pi}{\arccos(1 - d/2f)} \quad (10)$$

where N must clearly be an integer, otherwise we would take several cycles of 2π . The two linearly independent eigenvectors belonging to the eigenvalues (9) show to be, apart from an arbitrary constant factor,

$$\begin{pmatrix} -d \\ 1 - \hat{\gamma}_+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -d \\ 1 - \hat{\gamma}_- \end{pmatrix}. \quad (11)$$

With these eigenvectors we form the matrix

$$\Gamma_t = \begin{pmatrix} -d & -d \\ 1 - \hat{\gamma}_+ & 1 - \hat{\gamma}_- \end{pmatrix} \quad (12)$$

and its inverse matrix

$$\Gamma_t^{-1} = \frac{1}{d(\hat{\gamma}_- - \hat{\gamma}_+)} \begin{pmatrix} (1 - \hat{\gamma}_-) & d \\ -(1 - \hat{\gamma}_+) & -d \end{pmatrix}. \quad (13)$$

The matrix Γ as defined in (1) is not symmetrical but its determinant is unity and its eigenvalues are distinct. Therefore, by applying a canonical diagonalization^{14, 15} we obtain its n -th power

$$\Gamma^n = \Gamma_t \begin{pmatrix} \hat{\gamma}_+^n & 0 \\ 0 & \hat{\gamma}_-^n \end{pmatrix} \Gamma_t^{-1} \quad (14a)$$

or written explicitly

$$\Gamma^n = \frac{1}{\sin \gamma} \begin{pmatrix} \sin n\gamma - \sin(n-1)\gamma & d \sin n\gamma \\ \frac{2}{d} (\cos \gamma - 1) \sin n\gamma & \sin(n+1)\gamma - \sin n\gamma \end{pmatrix}. \quad (14b)$$

¹⁰ M. BORN and E. WOLF, Principles of Optics, Pergamon Press, London, New York, Paris, Los Angeles 1959, pp. 108–201.

¹¹ P. DRUDE, The Theory of Optics, Dover Publications Inc., New York 1959, pp. 14–72.

¹² $d/f < 0$ means that the focal length is negative (divergent lenses).

¹³ G. D. BOYD and H. KOGELNIK, Bell System Techn. J. **41**, 1347 [1962].

¹⁴ H. MARGENAU and G. M. MURPHY, The Mathematics of Physics and Chemistry, D van Nostrand Company, Inc., New York 1956, pp. 319–322.

¹⁵ R. ZURMÜHL, Matrizen und ihre technischen Anwendungen, Springer-Verlag, Berlin-Göttingen-Heidelberg 1958, pp. 158–166.

For a beam waveguide with n periodically spaced thin lenses the matrix (14) completely describes the optical image formation when the initial values (ϱ_0, ϱ_0') are specified. The radius and slope behind the n -th length are determined from (5), using (14b).

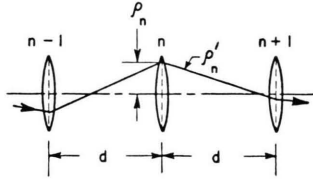


Fig. 1. A beam waveguide consisting of equally-spaced thin lenses. The path indicates a paraxial ray of light.

This analysis is illustrated by Fig. 2. We have schematically traced a beam of light as it propagates through different beam waveguides. Each beam waveguide consists of a periodic sequence of thin lenses but spaced differently. In Fig. 2a the lenses are degenerated into plane parallel plates, infinitely thin. We notice that the ray structure reproduces itself after N lenses where N is given by (10). Except for $N=2$ Fig. 2b, each section consisting of N lenses represents a telescopic system. A telescopic system is characterized by having both focal planes at infinity. Rays impinging parallel to the axis from the object-space pass from the system into the image-space, parallel to the axis. Thus, if we assume that the initial slope ϱ_0' is identical with zero the telescopic systems are determined by the condition that the matrix element I_{21}^n of (14b) vanishes. Because of their convenient properties it is quite likely that beam waveguides representing a telescopic system will play a significant role in future transmission lines.

2. Optical Analogy between a Thin Lens and a Spherical Mirror

In geometrical optics the familiar law

$$\frac{f}{\zeta} + \frac{f'}{\zeta'} = 1 \quad (15)$$

covers the image formation of a thin lens as well as that of a spherical mirror. The primed and unprimed notation is used to distinguish between the image and the object space. The distance of the object and the image from the unit plane are termed ζ and ζ' . The focal lengths of a thin lens are given by the

inverse relation

$$\frac{1}{f} = \frac{1}{f'} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad (16)$$

where n is the index of refraction of the medium of the lens. The radii of curvature R_1 and R_2 of the lens towards the object and image space are to be considered positive if the sphere is convex towards the incident light, otherwise negative. With this same sign convention the focal lengths of the spherical mirror are

$$f = -R/2 \quad \text{and} \quad f' = R/2. \quad (17)$$

The two principle foci coincide in the mid-point between the vertex and the center of curvature.

These two kinds of optical systems differ from one another only in the signs of the focal lengths f and f' . The lens represents a concurrent system, i. e., if the object moves from left to right, the image likewise moves from left to right. The mirror represents a contracurrent system, i. e., if the object moves from left to right, the image moves from right to left, as it appears from NEWTON's formula $z z' = f f'$. By the usual convention, the concave mirror corresponds to a convergent system, e. g., a double-convex lens; on the other hand a convex mirror corresponds to a divergent system, e. g., a double concave lens¹¹.

In light of this analogy we can replace each lens by a mirror of the same size with a radius of curvature twice the focal length. A beam of light which would propagate through a beam waveguide thus bounces back and forth between two mirrors. This picture reveals the close relationship between a beam waveguide and a resonator system at optical frequencies. Let us assume that the lenses are perfectly transmitting and that the mirrors are perfectly reflecting. Then the remaining losses of both systems, namely the diffraction losses, are equal. Hence, the image formation and the diffraction losses of a beam waveguide may be studied by using the simpler formulation which is appropriate for a mirror system. This analogy will be the basis of the theory presented for beam waveguides.

3. Integral Theorem for Successive Reflections

We shall investigate mirror systems which have no side-wall boundaries and in which the medium is assumed to fill all space. Such a system is shown in Fig. 3. The mirrors S^l and S^r , which are assumed to

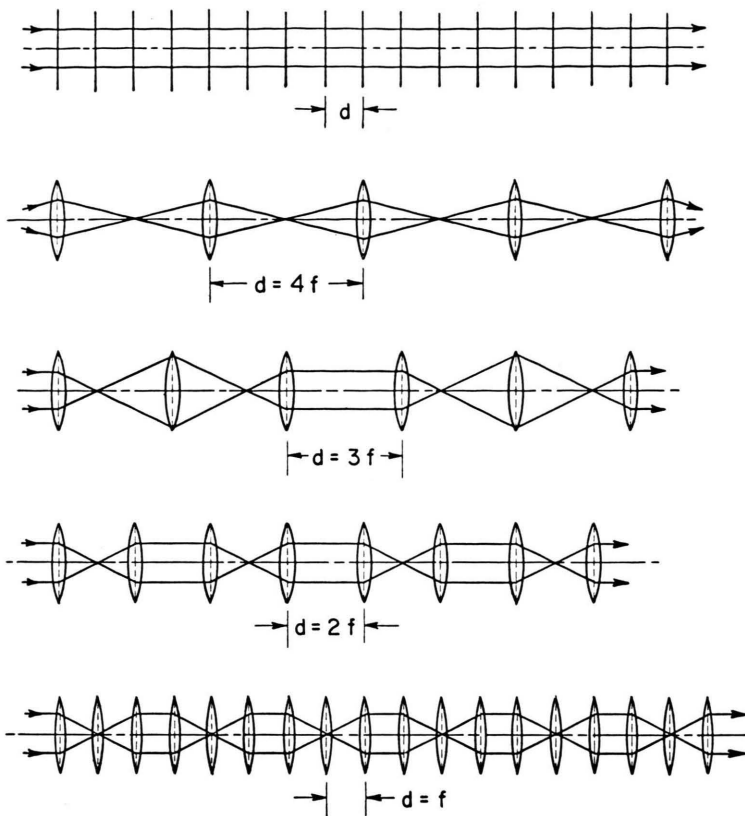


Fig. 2.

Fig. 2. Sections of beam waveguides whose thin lenses are spaced differently. The paths indicate a beam of light as it propagates along the beam waveguide. a) Beam waveguide of the iris type: $N=1$ ($f \rightarrow \infty$), b) $d=4f$ and $N=2$, c) $d=3f$ and $N=3$, d) $d=2f$ and $N=4$, e) $d=f$ and $N=6$.

Fig. 4. Illustrating the resonator systems with identical spherical reflectors which correspond to the various beam waveguides shown in Fig. 2. Only one path indicating a paraxial ray of light is traced. The systems (b) to (e) merely differ by the separation of the mirrors as stated in each picture. The plane parallel interferometer of (a) represents the limiting case as $R \rightarrow \infty$ for d being fixed. The dotted lines in (c) indicate a second "eigenmode" for a complete round-trip between the reflectors.

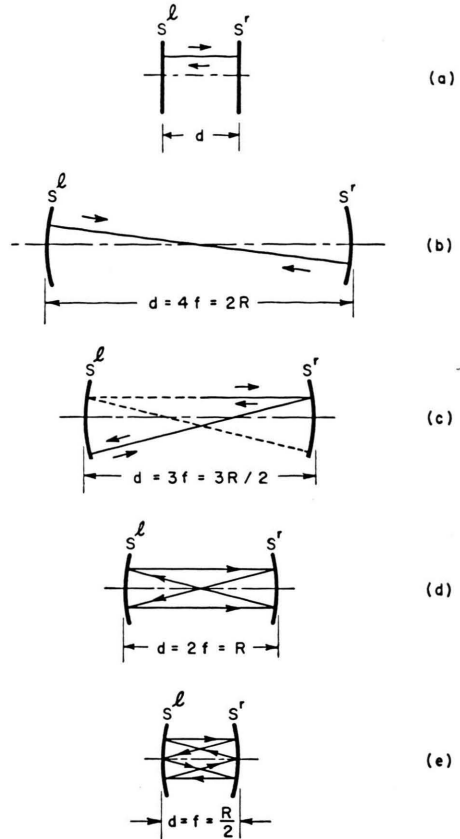


Fig. 4.

be perfectly conducting and to be uniformly reflecting, are, in general, of different sizes and/or shapes (the superscripts l and r refer to "left" and "right"

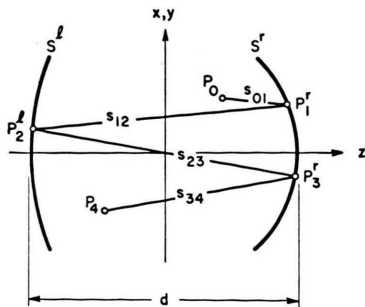


Fig. 3. Illustrating the integral theorem (18).

respectively). The dimensions of the mirrors are assumed to be large compared to the wavelength.

The author¹⁵ has derived an integral theorem for an optical wave which bounces n times between two mirrors; the spacing of these mirrors must not be equal in successive bounces. This integral theorem based upon the HUYGENS-FRESNEL Principle accounts not only for the rectilinear propagation of light of very short wavelength, but also for the law of reflection and for certain diffraction phenomena. It may be formulated as follows: "Let the monochromatic wave function ψ be a solution of the HELMHOLTZ

¹⁵ H. LOTSCH, A Scalar Resonator Theory for Optical Frequencies, to be published in Opt. Acta.

Equation $(\nabla^2 + k^2)\psi = 0$, whose first and second-order partial derivatives are continuous within and on the surfaces S^l and S^r , and let¹⁷

$$\psi(P_{n+1}) = A \left(\frac{i}{\lambda} \right)^n \int_{S_1^r} \cdots \int_{S_n^l} \frac{\exp \{ i k \sum_{j=0}^n s_{jJ} \}}{\prod_{j=0}^n s_{jJ}} \cdot \prod_{j=0}^{n-1} \cos(n_{j+1}^l, s_J) \cdot dS_n^l \cdots dS_1^r \quad (18)$$

with $J \equiv (j, j+1)$ and $\bar{J} \equiv (j+1, j)$

where s_J is the distance between two points P_j and P_{j+1} , $k = 2\pi/\lambda$ is the propagation constant for the wavelength λ , A is a constant amplitude factor, and n_{j+1}^l is the inward normal at the point P_{j+1} to the right and left surface respectively. Then the value of (18) is $\psi(P_{n+1})$ or zero, according to whether P_{n+1} lies inside or outside of the resonator space.¹⁸ This theorem expresses $\psi(P_{n+1})$ in terms of GREEN's functions for perfectly conducting plane surfaces. In the range of optical wavelengths an electromagnetic wave essentially propagates along a rectilinear path. Therefore, to obtain a first-order approximation for a slightly curved mirror, we may assume that (18) applies locally and we may think of replacing this slightly curved surface by a "polyhedron" of plane elements. Consequently, if the radius of curvature of a spherical reflector is very large, we may use (18) as an acceptable approximation.

The question remains whether or not these assumptions are also physically justifiable. The answer is that they are valid approximations only for sufficiently small wavelengths, and the results must be checked against experiment.

4. General Formulation

In Section 2 we have elaborated the optical analogy between a thin lens and a spherical mirror^{18a}. Let us now construct a "Gedanken-Experiment" and think of replacing each lens of the beam waveguide by a spherical mirror with a radius of curvature of twice the focal length. The beam of light which would propagate through the beam waveguide bounces back and forth between the mirrors in this optically analogous model. To illustrate the model, Fig. 4

shows the resonator systems which correspond to the periodic sequences of thin lenses as shown in Fig. 2. In each case only one ray indicates the beam of light.

The simple scalar formulation of the HUYGENS-FRESNEL Principle can be applied to an electromagnetic problem according to KIRCHHOFF. We must assume that the electric field is very nearly transverse and, for instance, uniformly polarized in the y -direction. Then, the scalar wave function of (18) can be identified with this transverse component of the electric field $E_y \equiv \psi$; the appropriate boundary conditions have already been taken into account.

Let us assume that a field distribution $E_y(S_0^m)$ is given over a mathematical surface^{18b} S^m , where S^m is completely arbitrary within the assumption $k s_{01} \gg 1$. We may think of moving the source point P_0 over this surface S_0^m where the amplitude A changes its value according to the given distribution $E(S_0^m)$. We must sum up the contributions from all the sources on S_0^m to obtain the field distribution $e(S_{n+1}^m)$ on the mathematical surface S_{n+1}^m after n reflections, i. e., beyond the n -th lens. This is conveniently expressed in the form of an integral equation

$$\kappa e(S_{n+1}^m) = \int_{S_0^m} E(S_0^m) K(S_{n+1}^m, S_0^m) dS_0^m \quad (19)$$

with the corresponding kernel deduced from (18)

$$K(S_{n+1}^m, S_0^m) = \frac{\exp \{ i(n \mp 1)\pi/2 \}}{\lambda^{(n+1)}} \cdot \int_{S_1^r} \cdots \int_{S_n^l} \frac{\exp \{ i k \sum_{j=0}^n s_{jJ} \}}{\prod_{j=0}^n s_{jJ}} \cdot \prod_{j=0}^{n-1} \cos(n_{j+1}^l, s_J) \cdot dS_n^l \cdots dS_1^r \quad (20)$$

where J and \bar{J} stand for $(j, j+1)$ and $(j+1, j)$, respectively, and the extra λ is absorbed in the constant factor κ . We direct attention to the fact that the formulation of the integral equation introduces a phase shift of ± 90 degrees, which is not due to the physical problem. Thus, we compensate for this extra phase shift by arbitrarily introducing a phase factor¹⁶ $e^{\mp i\pi/2}$. We note that in this analogous model each reflection changes the phase of κ by π .

For many applications we may neglect the variation of the inclination factors over the reflecting surfaces and set $\cos(\cdots) \approx 1$ since only small angles

¹⁷ The indices of S correlate surface and surface element in the sequence of integration.

^{18a} See also: H. LOTSCH, Phys. Letters **11**, 221 [1964].

^{18b} We note that no particular boundary conditions need be considered on a mathematical surface.

are involved. Thus, the kernel (20) reduces to

$$K(S_{n+1}^m, S_0^m) = \frac{\exp\{i(n+1)\pi/2\}}{\lambda^{(n+1)}} \cdot \int_{S_1^1} \cdots \int_{S_n^1} \frac{\exp\{i k \sum_{j=0}^n s_j\}}{\prod_{j=0}^n s_j} dS_n^1 \cdots dS_1^1. \quad (21)$$

Equation (19) with the kernel (20) or (21) represents a homogeneous FREDHOLM integral equation of the first kind, κ is a constant factor. This integral equation represents a traveling-wave type solution of the HELMHOLTZ Equation which satisfies MAXWELL's Equations along the boundaries of the reflecting surfaces. For any initial distribution $E(S_0^m)$ over the mathematical surface S_0^m (19) yields the distribution $e(S_{n+1}^m)$ over the mathematical surface S_{n+1}^m behind the n -th lens. We shall demonstrate later that (19) is completely consistent with the geometrical-optics solution in the zero-wavelength limit.

5. Diffraction Losses of the Beam Waveguide

In Section 1 we have demonstrated that for a separation d belonging to the range (8), the beam of light changes its diameter periodically as it propagates along the beam waveguide. GAUSSIAN or geometrical optics represents the limiting case for HUYGENS' undulatory theory at zero-wavelength, and hence, both these theories must show the same periodicity N given by (10). Let us define an eigenmode as an energy distribution which when launched from a mathematical surface is reproduced on the corresponding mathematical surface after having passed through N lenses. In the optically analogous model of a resonator system these mathematical surfaces coincide. For such a self-consistent field distribution we set $e(S_{n+1}^m) \equiv E(S_{n+1}^m)$ in (19) and obtain the homogeneous FREDHOLM integral equation of the second kind

$$\kappa_{pq} E_{pq}(S_{N+1}^m) = \int_{S_0^m} E_{pq}(S_0^m) K(S_{N+1}^m, S_0^m) dS_0^m \quad (22)$$

with the corresponding kernel from (21)

$$K(S_{N+1}^m, S_0^m) = \frac{\exp\{i(N+1)\pi/2\}}{\lambda^{(N+1)}} \cdot \int_{S_1^1} \cdots \int_{S_N^1} \frac{\exp\{i k \sum_{j=0}^N s_j\}}{\prod_{j=0}^N s_j} dS_N^1 \cdots dS_1^1 \quad (23)$$

where J stands for $(j, j+1)$. The indices p and q specify the order of the eigenmode; κ_{pq} and E_{pq} are the eigenvalues and eigenfunctions of the pq -th eigenmode.

From energy balance considerations we may deduce the relation

$$\alpha_{pq} = 1 - |\kappa_{pq}|^2 \quad (24a)$$

for the fractional diffraction losses of the pq -th eigenmode. If $N > 1$ we may find in some cases that an eigendistribution is already reproduced, but inverted, behind the $(N/2)$ -th lens as, for instance, when $d = 2f$, Fig. 2d¹⁹. When formulating the integral equation (22) for such a section of an eigenmode, the fractional diffraction loss of the pq -th eigenmode is given by

$$\alpha_{pq} = 1 - |\kappa_{pq}|^{2\gamma} \quad (24b)$$

where here γ represents the number of identical sections composing an eigenmode.

Numerical values of the fractional losses are given¹⁹ for the lowest-order eigenmodes of the FABRY-PEROT Interferometer, the confocal and the spherical resonator systems. These resonator systems correspond to the beam waveguides of the iris type and of the types $d = 2f$ and $4f$. The general-type eigenmode of the confocal resonator system is found to have the lowest losses. As is pointed out¹⁹, this eigenmode is associated with the telescopic system formed by the confocally spaced lenses.

An arbitrary initial distribution $E(S_0^m)$ may be approximated in terms of eigenfunctions $E_{pq}(S_0^m)$. From this superposition we can determine the corresponding fractional diffraction losses. We note that the arbitrary distribution $E(S)$ changes as it propagates along the beam waveguide since the various eigenmodes E_{pq} of this superposition have different losses. If the beam waveguide is very long only the lowest-order eigenmodes will survive and arrive at the terminal.

Finally, we wish to discuss a very special initial field distribution. In a laser device with a symmetrical resonator system we expect the mode patterns over both the mirrors to be identical. These patterns, which as a whole seem to be self-consistent from transit to transit, may be considered a special super-

¹⁹ H. LOTSCH, Multimode Resonators with a Small FRESNEL Number (Lowest-Order Eigenmodes), Z. Naturforschg. **20a** [1965], in press.

position according to HEFFNER²⁰. This superposition consists of N components on each mirror which belong to N eigenmodes uniformly displaced by n/N along the direction of propagation. Figure 5 illustrates this self-consistent distribution using the optically analogous model of a periodic sequence of thin lenses $d = 3f$ apart. Such a superposition viewed as a whole does not seem to follow the laws describing the image formation in geometrical optics. We must, however, direct attention to the most significant fact that each component of this superposition obeys the classical laws. We believe that this physical model has not yet been discovered by other investigators. It apparently removes the general confusion concerning the validity of classical laws when applied to a laser beam. We easily convince ourselves that if such a self-consistent distribution propagates along a beam waveguide, its fractional diffraction losses per transit to the following lens are approximately identical with the fractional diffraction loss of the corresponding eigenmode.

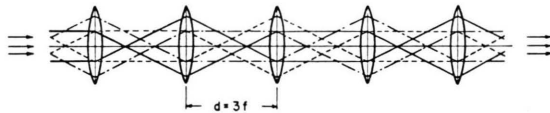


Fig. 5. A section of a beam waveguide whose lenses are periodically spaced by $d = 3f$. Three eigenmodes are shown which are displaced by $1/3$ and $2/3$ of their length. This superposition illustrates the distribution which is reproduced from lens to lens.

6. Optimum Matching

In the preceding section we have demonstrated that the eigendistributions represented by the eigenfunctions of (22) are reproduced within a constant amplitude factor after having passed through N lenses. These eigendistributions form a set of stationary solutions, the associated losses are determined by (24). Let us assume that a laser device is constructed from two equal spherical mirrors with a radius of curvature R a distance $d = \text{const} \times (R)$ apart. This laser device is matched to a beam waveguide with identical, thin lenses $d = \text{const} \times (2f) \equiv \text{const} \times (R)$ apart. If the laser operates in a quasi steady-state condition, the mode patterns are composed only of the lowest-order eigenmodes. From the optical analogy between a resonator system and a beam wave-

guide we know that both these corresponding types of systems are described by the same set of eigenmodes. Therefore, if these systems are matched, the mode pattern associated with an eigenmode is reproduced within a constant amplitude factor after having passed through a multiple of N lenses, and the diffraction losses are a minimum.

7. Application of the General Theory

As an application of the general theory we now investigate the beam waveguide which consists of a periodic sequence of thin lenses a distance $d = 3f$ apart. We are concerned only with self-consistent field distributions; the periodicity is determined from (10) to be $N = 3$. This beam waveguide corresponds to a resonator system with equal spherical mirrors a distance $d = 1.5R$ apart. These mirrors are assumed to have a uniform reflectivity, to be equal in size and, for convenience, to be square when viewed in the z -direction. All dimensions are large compared to the wavelength and the field is assumed to be linearly polarized in the y -direction. Figure 6 illustrates the formulation of the problem under investigation. We have chosen the plane of symmetry as the mathematical surface. The distances s_{01}, \dots, s_{34} , which the paraxial wavetrains travel, vary only by a small amount and may be expressed in terms of the separation d of the two mirrors, except in the exponential phase term of (23). If x and y are small compared to d we can show that

$$s_{01} \approx \frac{d}{2} + \frac{x_0^2 - 2x_0x_1 + y_0^2 - 2y_0y_1}{d} + \frac{x_1^2 + y_1^2}{4d}, \quad (25)$$

$$s_{12} \approx d - \frac{x_1x_2 + y_1y_2}{d} - \frac{1}{4d}(x_1^2 + x_2^2 + y_1^2 + y_2^2) \quad (26)$$

and the expressions for s_{23} and s_{34} are easily deduced from (26) and (25).

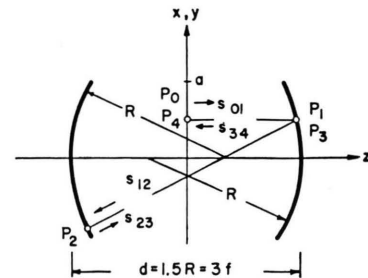


Fig. 6. Analogous model for a beam waveguide with $d = 3f = 1.5R$ to illustrate the derivation of (28).

²⁰ H. HEFFNER, Electrical Engineering Department, Stanford University, Stanford, California (private communication).

Let us assume first that the field pattern of an eigenmode can be written in product form

$$E_y(x, y) = E_0 X_p(x) Y_q(y) \quad (27)$$

where E_0 is a constant amplitude factor, $X(x)$ is a function of x only, and $Y(y)$ is a function of y only, and secondly that the amplitude coefficient κ can also be expressed in product form. Since there are no cross products between x and y , we can separate the integral equation (22) into a product of two integral equations identical in form. One of these integral equations depends only upon x and the other one only upon y . Henceforth, we investigate merely the x -dependent integral equation and use the same results for the y -dependent one. We introduce the dimensionless variable x/a for x . Then the resulting x -dependent integral equation is

$$\kappa_p X_p(x_4) = \int_{-\infty}^{+\infty} K(x_4, x_0) X_p(x_0) dx_0 \quad (28)$$

with the corresponding kernel from (23) using (25) and (26)

$$\begin{aligned} K(x_4, x_0) = & 2 F^2 \cdot \exp \{i(3 k d + \pi)/2\} \\ & \cdot \exp[i 2 \pi F(x_4^2 - x_0^2)] \int_{-1}^{+1} \exp[i 4 \pi F x_3(x_0 - x_4)] \\ & \cdot \int_{-1}^{+1} \exp[i \pi F(x_1 + x_3 - 2 x_0)^2] \\ & \cdot \int_{-1}^{+1} \exp[-i \pi F(x_2 + x_1 + x_3)^2] \cdot dx_2 dx_1 dx_3 \end{aligned} \quad (29)$$

where the FRESNEL Number F is defined by

$$F = a^2/(\lambda d) . \quad (30)$$

The integral equation (28) with its kernel (29) appears too involved to expect a solution in terms of ordinary functions. Therefore, it must be treated by numerical methods. To demonstrate the features of the theory presented in the zero-wavelength limit, ($F \rightarrow \infty$), i. e., in the geometrical-optics solution, we can evaluate the FRESNEL Integrals using the method of stationary phase (see e. g. ¹⁶ or ²¹). According to this method, contributions from parts of the range of integration near a point of stationary phase will be nearly in phase and add up, whereas those from other parts will interfere destructively. Thus, the x_1 and x_2 integrals of (29) yield non-zero

values for a large FRESNEL Number if and only if the conditions $x_1 \approx (2 x_0 - x_3)$ and $x_2 \approx -(x_1 + x_3)$ are satisfied. Attention is directed to these conditions since they represent the law of reflection at the points P_1 and P_2 , specified by x_1 and x_2 . Finally we obtain the eigenvalue

$$\kappa_{pq} = \kappa_p \kappa_q = -e^{i 3 k d} \quad \text{as } F \rightarrow \infty . \quad (31)$$

The diffraction losses vanish according to (24). The minus sign appears in (31) since the linearly polarized electromagnetic wave is reflected three times by a perfectly conducting surface in the optically analogous model of a resonator system. Hence, we have demonstrated that this theory for beam waveguides based upon HUYGENS' undulatory theory is completely consistent with the classical laws in the zero-wavelength limit.

8. Conclusion

An elementary theory for beam waveguides has been developed. The image formation of a beam waveguide can be described by a matrix equation based upon geometrical optics, or by an integral equation based upon both HUYGENS' undulatory theory and the optical analogy between a thin lens and a spherical mirror. Diffraction losses were defined for an eigenmode, i. e., for a field distribution which is reproduced after having passed through N lenses. This number N represents the periodicity of a beam waveguide. A special distribution is discussed which reproduces itself from lens to lens. According to HEFFNER this distribution consists of a superposition of N eigenmodes uniformly displaced in the direction of propagation. It is suggested that the laser resonator be matched with its optically analogous beam waveguide for an optimum transmission.

The general theory when applied to a beam waveguide under investigation yields results which are consistent with the classical laws in the zero-wavelength limit.

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²¹ H. JEFFREYS and B. S. JEFFREYS, *Method of Mathematical Physics*, Cambridge University Press, London 1956, pp. 506–507.

Prof. H. HEFFNER of Stanford University, Prof. C. H. PAPAS of the California Institute of Technology, and Prof. Dr. E.h. F. SCHROETER of University of Bonn and Telefunken AG, Germany.

Appendix

Geometrical Theory for a Beam Waveguide Consisting of a Periodic Sequence of Two Unequal Thin Lenses

Fig. 7 shows a beam waveguide consisting of a periodic sequence of two unequal thin lenses but spaced equal distances d apart. In order to apply the derivation of Section 1, we consider the even-numbered lenses $n, n+2, n+4$, etc., as having an equal focal length $f_2 = 1/C_2$ and correspondingly the odd-numbered lenses $n+1, n+3$, etc., an equal focal length $f_1 = 1/C_1$. Combining a pair of adjacent lenses even and odd numbered to one unit, we obtain with the notation of Section 1

$$\begin{pmatrix} \varrho_{n+2} \\ \varrho'_{n+2} \end{pmatrix} = \Sigma \begin{pmatrix} \varrho_n \\ \varrho'_n \end{pmatrix} \quad (\text{A } 1)$$

where the matrix Σ is given by

$$\Sigma = \begin{pmatrix} (1-C_1 d) & d(2-C_1 d) \\ (C_1 C_2 d - C_1 - C_2) & (1-C_1 d - 2 C_2 d + C_1 C_2 d^2) \end{pmatrix}. \quad (\text{A } 2)$$

Acceptable solutions are obtained if and only if $|\text{trace } \Sigma| \leq 2$ which yields the regions of stability or low-loss regions assuming $f_1 \leq f_2$

$$0 \leq d \leq 2 f_1 \quad (\text{A } 3 \text{ a})$$

$$\text{and} \quad 2 f_2 \leq d \leq (2 f_1 + 2 f_2). \quad (\text{A } 3 \text{ b})$$

For a separation d belonging to these low-loss regions the desired periodicity is derived from (10)

$$N = \frac{4 \pi}{\arccos[1 - d(1/f_1 + 1/f_2) + d^2/2 f_1 f_2]} \quad (\text{A } 4)$$

where the extra factor of 2 compensates for the fact that one unit consists of a pair of lenses.

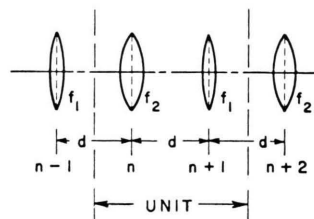


Fig. 7. Beam waveguide consisting of a periodic sequence of lenses of alternating focal length. A pair of lenses (f_1 and f_2) form a basic unit.

The ν -th power of the matrix Σ of (A 2) is similarly obtained by a canonical diagonalization

$$\Sigma^\nu = \frac{1}{\sin \sigma} \begin{pmatrix} (1-C_1 d) \sin \nu \sigma - \sin(\nu-1) \sigma & d(2-C_1 d) \sin \nu \sigma \\ (C_1 C_2 d - C_1 - C_2) \sin \nu \sigma & \sin(\nu+1) \sigma - (1-C_1 d) \sin \nu \sigma \end{pmatrix} \quad (\text{A } 5)$$

where the roots of the secular equation are re-defined by

$$\hat{\sigma}_+ = e^{i \sigma} \quad \text{and} \quad \hat{\sigma}_- = e^{-i \sigma}. \quad (\text{A } 6)$$

In the limit $f_1 \equiv f_2$ (A 5) is consistent with (14) noting that $n = 2 \nu$ since a basic unit here contains two lenses.